B.sc. part-I, paper-2

General equation of second degree

Theorem :- To show that general equation of second degree always represents a conic section & Find the condition under which the equation
$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
will represent a parabola, an ellipse or a hyperbola

proof :- Let the general equation of second degree is
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{— (1)}$$
Let the axes of co-ordinates be turned through an angle θ, keeping the origin unchanged then substituting $x\cos\theta - y\sin\theta$ and $x\sin\theta + y\cos\theta$ for x and y in (1) we get
$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)$$
$$(x\sin\theta + y\sin\theta) +$$

$b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta)$
$+ 2f(x\sin\theta + y\cos\theta) + c = 0$

or $x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$
$+ 2xy\{h(\cos^2\theta - \sin^2\theta) + (b-a)\sin\theta\cos\theta\}$
$+ y^2(a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta)$
$+ 2x(g\cos\theta + f\sin\theta) + 2y(f\cos\theta - g\sin\theta)$
$+ c = 0$ — (2)

Now we choose θ so that the coefficient of xy in (2) become zero. For this we have

$h(\cos^2\theta - \sin^2\theta) + (b-a)\sin\theta\cos\theta = 0$

$\therefore h\cos 2\theta - \frac{1}{2}(b-a)\sin 2\theta = 0$

or $\tan 2\theta = \dfrac{2h}{b-a}$ — (3)

Let the changed form of the equation (2) be

$Ax^2 + By^2 + 2Gx + 2Fy + C = 0$ — (4)

where $A = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$

Now the following case arise

Case (1) Let $A \neq 0$ and $B \neq 0$ The equation (4) may be written as

$$A\left(x^2+\frac{2Gx}{A}+\frac{G^2}{A^2}\right)+B\left(y^2+\frac{2Fy}{B}+\frac{F^2}{B^2}\right)$$
$$-\frac{G^2}{A}-\frac{F^2}{B}+C=0$$

or $A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{G^2}{A}+\frac{F^2}{B}-C$

$= K$ (say)

Shifting the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$ this equation becomes

$$Ax^2+By^2=K$$

i.e.
$$\frac{x^2}{\frac{K}{A}}+\frac{y^2}{\frac{K}{B}}=1 \quad \text{--- (5)}$$

If $\frac{K}{A}>0$ and $\frac{K}{B}>0$ then (5) represent an ellipse

If $\frac{K}{A}$ and $\frac{K}{B}$ be one positive and other negative then (5) represents a hyperbola.

If $\frac{K}{A}<0$ and $\frac{K}{B}<0$ the (5) represents an imaginary ellipse.

Case (ii) Let either A or B be zero and let it be A

then from (2) $By^2+2Gx+2Fy+C=0$

ie. $B(y+\frac{F}{B})^2+2G(x+\frac{C}{2G}-\frac{F^2}{2BG})=0$

Shifting the origin to $\left(-\frac{C}{2G}+\frac{F^2}{2BG^2}, -\frac{F}{B}\right)$

this equation becomes

$$By^2+2Gx=0$$

ie $$y^2=\frac{2G}{B}x$$

which represents a parabola.

~~problem~~ Another criteria

By rotating the axes through an angle θ, the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is transformed into the equation $Ax^2+By^2+2Gx+2Fy+C=0$

Hence by the theory of invariants

$$ab-h^2=AB$$

If either A or B is zero $ab-h^2=0$

we have shown above in case II that if A & B is zero the eqn represent a parabola

Hence the equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$

represents a parabola if $ab-h^2=0$

If neither A nor B is zero $ab-h^2\neq 0$

Thus if A and B be both +ve i.e. when $ab-h^2$ is +ve the equation

(4) evidently represents an ellipse (Case I)

The equation (4) represents a hyperbola if A and B are of opposite sign i.e $AB-h^2$ is negative

Thus the general equation $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents

(I) a parabola if $ab-h^2=0$

(II) an ellipse if $ab-h^2>0$

(III) a hyperbola if $ab-h^2<0$